Longitudinal Modes of Bunched Beams with Weak Space Charge

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Fermilab

Many thanks to I. Karpov, T. Argyropoulos and E. Shaposhnikova
for clarifications
and V. Lebedev for useful discussions.

ICFA Space Charge Workshop, Oct 25, 2022, Knoxville TN

AB

A LONGITUDINAL STABILITY CRITERION FOR BUNCHED BEAMS

F.J. Sacherer
CERN, Geneva, Switzerland

1973

$$N\left|\frac{Z}{n}\right|_{th} \propto \sigma_s^5$$

IEEE Transactions on Nuclear Science, Vol. NS-26, No. 3, June 1979

BUNCHES WITH LOCAL ELLIPTIC ENERGY DISTRIBUTIONS

A. Hofmann and F. Pedersen *)

1979

The threshold was reproduced, with different numerical factors, by several authors, including myself.

LONGITUDINAL BUNCH DYNAMICS IN THE TEVATRON

R. Moore, V. Balbekov, A. Jansson, V. Lebedev, K.Y. Ng, V. Shiltsev, C.Y. Tan

Fermi National Accelerator Laboratory, Batavia, IL, USA 60510

2003

DANCING BUNCHES AS VAN KAMPEN MODES

A. Burov, FNAL*, Batavia, IL 60510, U.S.A.

2011

AB

PHYSICAL REVIEW ACCELERATORS AND BEAMS **24**, 011002 (2021)

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Thresholds for loss of Landau damping in longitudinal plane

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(Received 13 November 2020; accepted 11 January 2021; published 27 January 2021)

The Landau damping mechanism plays a crucial role in providing single-bunch stability in the LHC, high-luminosity LHC, and other existing as well as previous and future circular hadron accelerators. In this paper, the thresholds for the loss of Landau damping (LLD) in the longitudinal plane are derived analytically using the Lebedev matrix equation (1968) and the concept of the emerged van Kampen modes (1983). We have found that for the commonly used particle distribution functions from a binomial family, the LLD threshold vanishes in the presence of the constant inductive impedance $\mathrm{Im}Z/k$ above transition energy. Thus, the effect of the cutoff frequency or the resonant frequency of a broadband impedance on beam dynamics is studied in detail. The findings are confirmed by direct numerical solutions of the Lebedev equation as well as using the Oide-Yokoya method (1990). Moreover, the characteristics, which are important for beam operation, as the amplitude of residual oscillations and the damping time after a kick (or injection errors) are considered both above and below the threshold. Dependence of the threshold on particle distribution in the longitudinal phase space is also analyzed, including some special cases with a nonzero threshold for $\mathrm{Im}Z/k = \mathrm{const}$. All main results are confirmed by macroparticle simulations and consistent with available beam measurements in the LHC.

$$\cancel{N\left|\frac{Z}{n}\right|_{th} \propto \sigma_s^5} \qquad N\left|\frac{Z}{n}\right|_{th} = 0$$

AB

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Longitudinal modes of bunched beams with weak space charge

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(Received 22 March 2021; accepted 18 May 2021; published 7 June 2021)

Longitudinal collective modes of a bunched beam with a repulsive inductive impedance (the space charge below transition or the chamber inductance above it) are analytically described by means of reduction of the linearized Vlasov equation to a parameterless integral equation. For any multipolarity, the discrete part of the spectrum is found to consist of infinite number of modes with real tunes, which limit point is the incoherent zero-amplitude frequency. In other words, notwithstanding the rf bucket nonlinearity and potential well distortion, the Landau damping is lost. Hence, even a tiny coupled-bunch interaction makes the beam unstable; such growth rates for all the modes are analytically obtained for arbitrary multipolarity. In practice, however, the finite threshold of this loss of Landau damping is set either by the high-frequency impedance roll-off or intrabeam scattering. Above the threshold, growth of the leading collective mode should result in persistent nonlinear oscillations.

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Particle Interaction

Beam particles interact with each other, driving collective instabilities.

This interaction is described by means of wake functions

$$F_{||} = e^2 \, W_0'(s)$$

and impedances

$$Z_0^{\|}(\omega) = \int_{-\infty}^{\infty} \frac{dz}{c} e^{-i\omega z/c} W_0'(z)$$

Space Charge or Inductance

$$W(s) \propto \delta(s)$$

$$Z_0^{\parallel}(\omega) = iZ_0 \frac{R\omega}{c\gamma^2}\left(\ln\frac{b}{a} + \frac{1}{2}\right)$$

$$n = \omega/\omega_0$$

$$\frac{Z}{n} = i\frac{Z_0}{\gamma^2}\left(\ln\frac{b}{a} + \frac{1}{2}\right)$$

SC is repulsive below transition, and attractive above it. The vacuum chamber inductance is described by similar formulas, but with opposite sign.

We'll focus on the repulsive case here (SC below transition or the inductance above)

Potential Well Distortion

Steady state problem:

$$H(z,p)=\frac{p^2}{2}+U(z);$$

$$U(z)=U_{\mathrm{rf}}(z)-\int_{\hat{z}_-}^{\hat{z}_+}\lambda(z')W(z-z')\mathrm{d}z'=U_{\mathrm{rf}}(z)+k\lambda(z);$$

$$I(H)=\frac{1}{\pi}\int_{z_-(H)}^{z_+(H)}\sqrt{2(H-U(z))}\mathrm{d}z;$$

$$\lambda(z)=2\int_{U(z)}^{\hat{H}}\frac{F(I(H))}{\sqrt{2(H-U(z))}}\mathrm{d}H;$$

$$2\pi\int_0^{\hat{I}}F(I)\mathrm{d}I=\int_{\hat{z}_-}^{\hat{z}_+}\lambda(z)\mathrm{d}z=1\,.$$

$$k=-\frac{2Nr_0\eta\,\omega_{\mathrm{rf}}^3}{\gamma\,c\,\Omega_0^2}\,\frac{\Im Z_n}{n\,Z_0}$$

AB

The Jeans-Vlasov Equation (JVE)

KEK Preprint 90- 10
April 1990
A

Longitudinal Single-Bunch Instability in Electron Storage Rings

Katsunobu Oide

and

Kaoru Yokoya

KEK, National Laboratory for High Energy Physics

Oho, Tsukuba, Ibaraki 305, Japan

HB2010

FERMILAB-CONF-10-376-AD

VAN KAMPEN MODES FOR BUNCH LONGITUDINAL MOTION

A. Burov, FNAL*, Batavia, IL 60510, U.S.A.

PAC’11

FERMILAB-CONF-11-121-AD

DANCING BUNCHES AS VAN KAMPEN MODES

A. Burov, FNAL*, Batavia, IL 60510, U.S.A.

Van Kampen approach: eigensystem of JVE

$$\left[\omega^2 - m^2\Omega^2(I)\right] f_m(I) =$$
$$-2m^2\Omega(I)F'(I)\sum_{n=1}^{\infty}\int dI' V_{mn}(I,I') f_n(I');$$

$$V_{mn}(I,I') =$$
$$-\frac{2}{\pi}\int_0^{\pi} d\varphi \int_0^{\pi} d\varphi' \cos(m\varphi)\cos(n\varphi') W(z(I,\varphi) - z(I',\varphi'))$$

$$V_{mn}(I,I') = -2\,\mathrm{Im}\int_0^{\infty} dq \frac{Z(q)}{q} G_m(q,I) G_n^*(q,I');$$
$$G_m(q,I) \equiv \int_0^{\pi} \frac{d\varphi}{\pi}\cos(m\varphi)\exp\left[iqz(I,\varphi)\right].$$

If the wake is small enough, the weak headtail approximation, $m = n$, is justified.

Continuous and discrete spectra

Is the beam stable, if the frequencies are all real? Not necessarily.

Generally, the spectrum consists of two parts, continuous and discrete (van Kampen)

Van Kampen, N. G.
1955

Physica XXI
949–943

ON THE THEORY OF STATIONARY WAVES IN PLASMAS

by N. G. VAN KAMPEN

Instituut voor theoretische natuurkunde, Rijksuniversiteit te Utrecht, Nederland

The continuous spectrum coincides with the incoherent one; its modes are singular functions, mostly located near the corresponding action. Their phase mixing corresponds to Landau damping.

The discrete modes lie outside the incoherent spectrum, they are described by non-singular analytical functions. If there is at least one discrete mode, Landau damping is lost.

Loss of Landau damping (LLD)

Landau damping is a natural immune system of beams against instabilities.

Without it, even a tiny ~~virus~~ coupled-bunch (CB) wake drives an instability.

Thus, Discrete Mode = LLD = CB Instability (saturated by nonlinearity).

Undamped coherent oscillations observed at Tevatron, RHIC, SPS and LHC can be attributed to the LLD.

PHYSICAL REVIEW ACCELERATORS AND BEAMS **24**, 011002 (2021)

Editors' Suggestion

Thresholds for loss of Landau damping in longitudinal plane

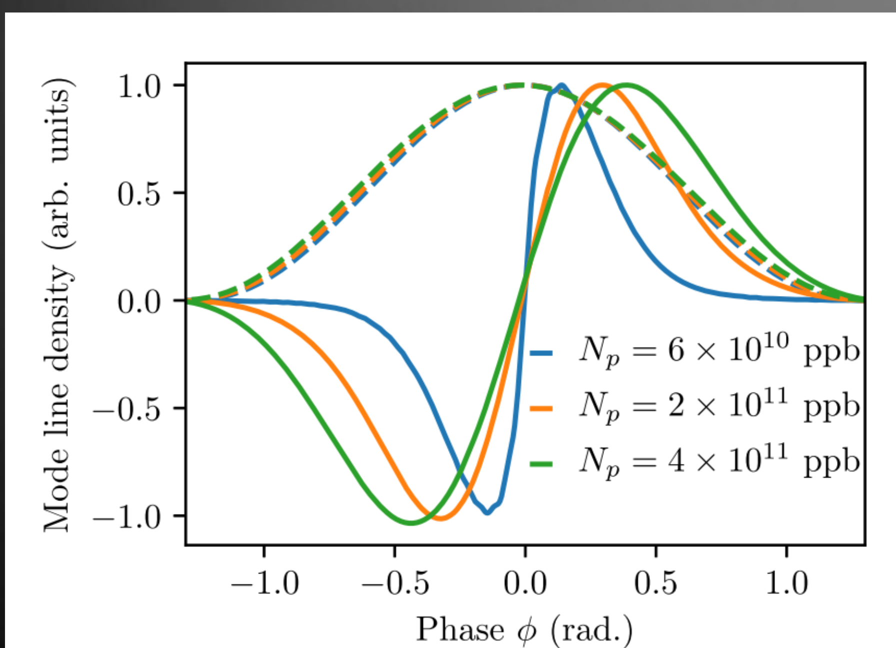
Ivan Karpov, Theodoros Argyropoulos, and Elena Shaposhnikova

CERN, CH 1211 Geneva 23, Switzerland

(Received 13 November 2020; accepted 11 January 2021; published 27 January 2021)

For $\quad F(I) \propto \left(1 - \frac{I}{\hat{I}}\right)^{\mu}, \qquad k_{th} \approx \dfrac{\pi\, \hat{b}^4}{32\, \mu\, (\mu+1)\, q_{max}} \quad \Leftrightarrow \quad \Delta b\, q_{max} \simeq 1$

That threshold set by the impedance roll-off q_{max} , suggests the mode size



$$\Delta b \simeq \frac{32k\, \mu\, (\mu+1)}{\pi\, \hat{b}^4} = 2k\, \frac{|F'|}{|\Omega'|} \equiv \alpha$$

Preliminary estimations

Let a central part of a bunch of a size a oscillate with a small amplitude $\tilde{z}$.

It yields the phase space density perturbation $f \simeq F'a\,\tilde{z}$

The line density perturbation $\rho \simeq f\,a \simeq F'a^2\,\tilde{z}$

And the reaction force $E \simeq k\rho/a \simeq k\,F'a\,\tilde{z}$

Corresponding to the coherent tune shift $\Delta\omega \simeq -k\,F'a/2 > 0$

It can be compared with the incoherent tune spread $\pm\delta\Omega = \pm|\Omega'|a^2/4$

If $\Delta\omega \geq \delta\Omega$, or $a \leq 2\,k\,|\,F'/\Omega'| \equiv \alpha$, Landau damping is lost.

For the collective oscillations with $a \leq \alpha$ there is no Landau damping.

AB

Four SC regimes

$\alpha \leq 1/q_{max}$ Insignificant SC. Landau damping is there (or IBS).

$1/q_{max} \leq \alpha \leq \sigma$ ↔ $k_{th} \leq k \leq 0.2\,\sigma^5$ Weak SC. Landau damping is lost.

$0.2\,\sigma^5 < k \ll 2\sqrt{2\pi}\,\sigma^3$ Medium SC. The relative tune depression is small

$k \simeq 2\sqrt{2\pi}\,\sigma^3$ Strong SC. The bucket is flattened, and the bunch profile is parabolic.

Weak SC, JVE reduction (1)

$$[\omega - \Omega(I)]f(I) = -F'(I)\int_0^{\hat{I}} K(I,I')f(I')\mathrm{d}I';$$

$$K(I,I') = -\frac{2}{\pi}\int_0^{\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\phi'\cos\phi\,\cos\phi'\,W(z(I,\phi)-z(I',\phi'))$$

$$K(I,I') = -2\Im\int_0^{\infty}\mathrm{d}q\frac{Z(q)}{q}G(q,I)\,G^*(q,I') = 2k\int_0^{\infty}\mathrm{d}q\,G(q,I)\,G^*(q,I')$$

$$G(q,I) = i\int_0^{\pi}\frac{\mathrm{d}\phi}{\pi}\cos\phi\,\exp[iqz(I,\phi)]\,.$$

$$z(I,\phi) = -\sqrt{2I}\cos\phi \equiv -b\cos\phi.$$

$$G(q,I) = \mathrm{J}_1(qb)$$

$$K(I,I') = \frac{4k}{\pi b_{\mathrm{min}}}\left[\mathrm{K}(u)-\mathrm{E}(u)\right] \geq 0;\ u \equiv b_{\mathrm{min}}^2/b_{\mathrm{max}}^2 \leq 1$$

Weak SC, JVE reduction (2)

$$f(I) = \frac{|F'|}{\Delta\omega + |\Omega'|I} \int_0^\infty K(I,I')f(I')\mathrm{d}I'$$

$$\Delta\omega = \omega - \Omega(0)$$

the coherent tune shift is with respect to what

$$I = b^2/2; \quad b = \alpha r; \quad \Delta\omega = |\Omega'|\alpha^2\nu/2; \quad f(\alpha^2 r^2/2) = \Phi(r)$$

scaling!

$$\alpha = 2k|F'|/|\Omega'|$$

$$\alpha = \frac{32k\,\mu\,(\mu+1)}{\pi\,\widehat{b}^4}$$ For $F(I) \propto \left(1 - \frac{I}{\hat{I}}\right)^{\mu}$;

$$\alpha = \frac{8\,k}{\pi\,\sigma^4}$$ For $F(I) \propto \exp(-I/\sigma^2)$.

Weak SC, JVE reduction (3)

$$\Phi(r)=\frac{1}{\nu+r^2}\int_0^\infty \mathrm{d}r' r' \Phi(r') Q(r,r')$$

discrete spectrum: $\nu > 0$

$$Q(r,r')=\frac{4}{\pi r_{\min}}\left[\mathrm{K}(u)-\mathrm{E}(u)\right];\ u\equiv r_{\min}^2/r_{\max}^2$$

note two singularities here

$$r_{\min} = \min(r,r'),\ r_{\max} = \max(r,r')$$

Big deal: the eigen-problem is reduced to a parameter-less equation!

$$\int_0^\infty \Phi_\beta(r)\Phi_\gamma(r) r \mathrm{d}r = \delta_{\beta\gamma};\quad \beta,\ \gamma = 1,2,3,\ldots\ .$$

Should we call that sort of solutions analytical?

Discrete Spectrum

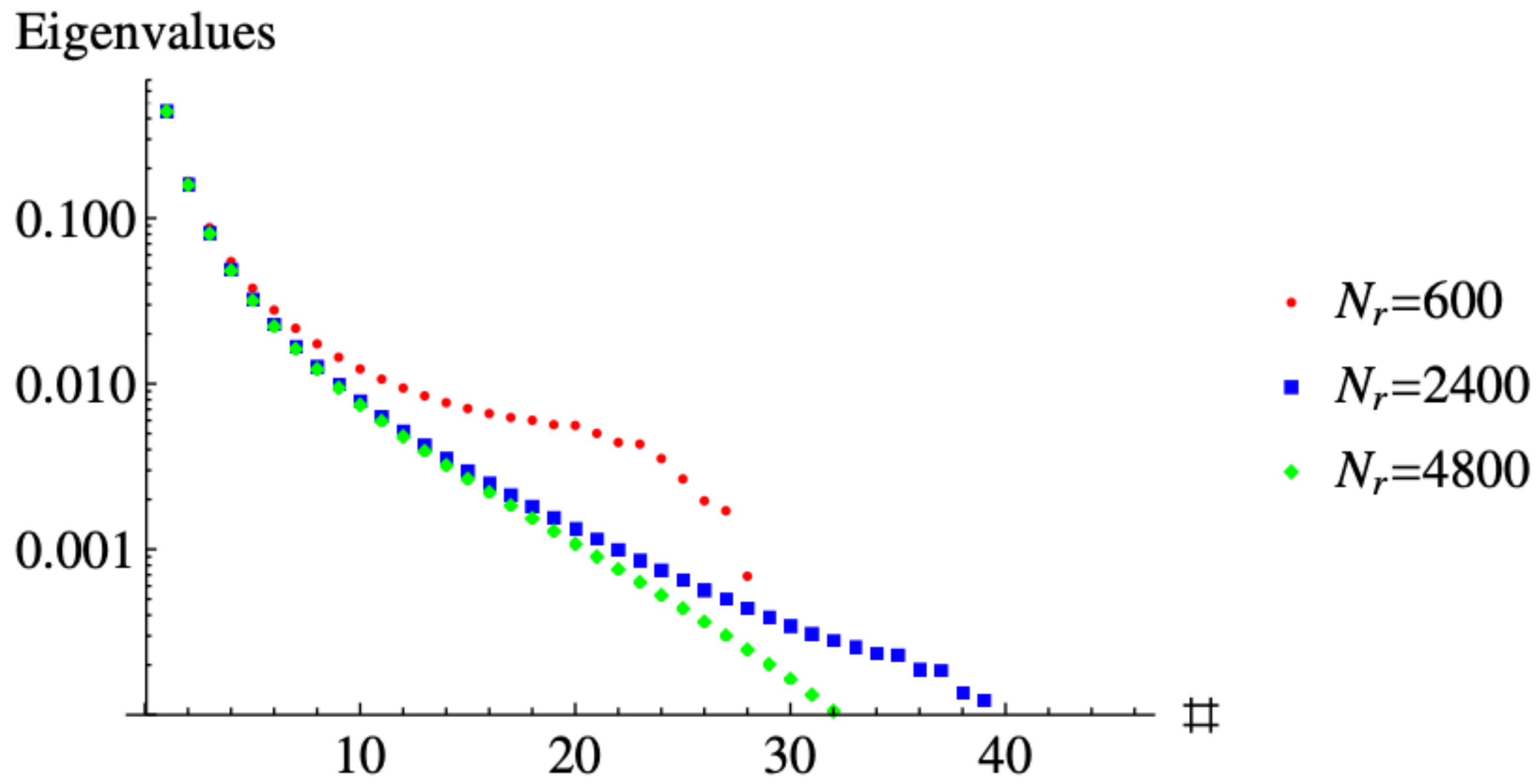


FIG. 1. Eigenvalues ν computed for three numbers of mesh points N_r, with the wake size $\sigma_w \simeq 0.005$ in the scaled units. The largest eigenvalues are $\nu = 0.44, 0.16, 0.08$. With denser mesh, the high-frequency tail of the spectrum changes, while its low-frequency part remains the same.

Eigenfunctions

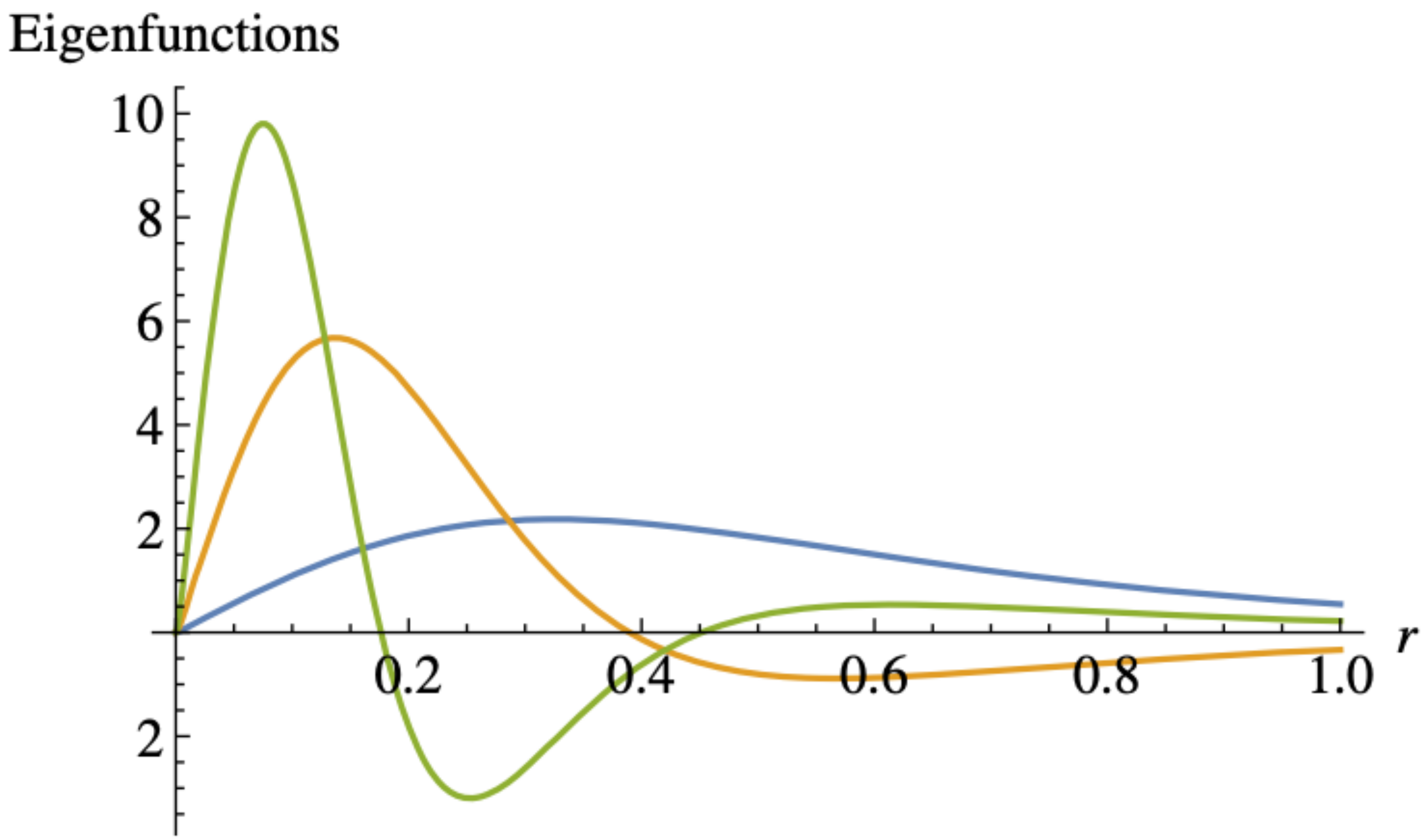


FIG. 3. Eigenfunctions $\Phi_\beta(r)$, for $\beta = 1, 2, 3$. The leading mode is blue, the next is yellow, and the third one is green. The eigenvalue of the leading mode is $\nu = 0.43$.

Eigenfunctions

$$\tilde{f} \propto \exp\left(-i\omega t + im\phi\right)$$

$$\rho(z) = \int \tilde{f}\mathrm{d}p = 2z\int_{|z|}^{\infty} \frac{\Phi(r)}{\sqrt{r^2 - z^2}}\mathrm{d}r$$

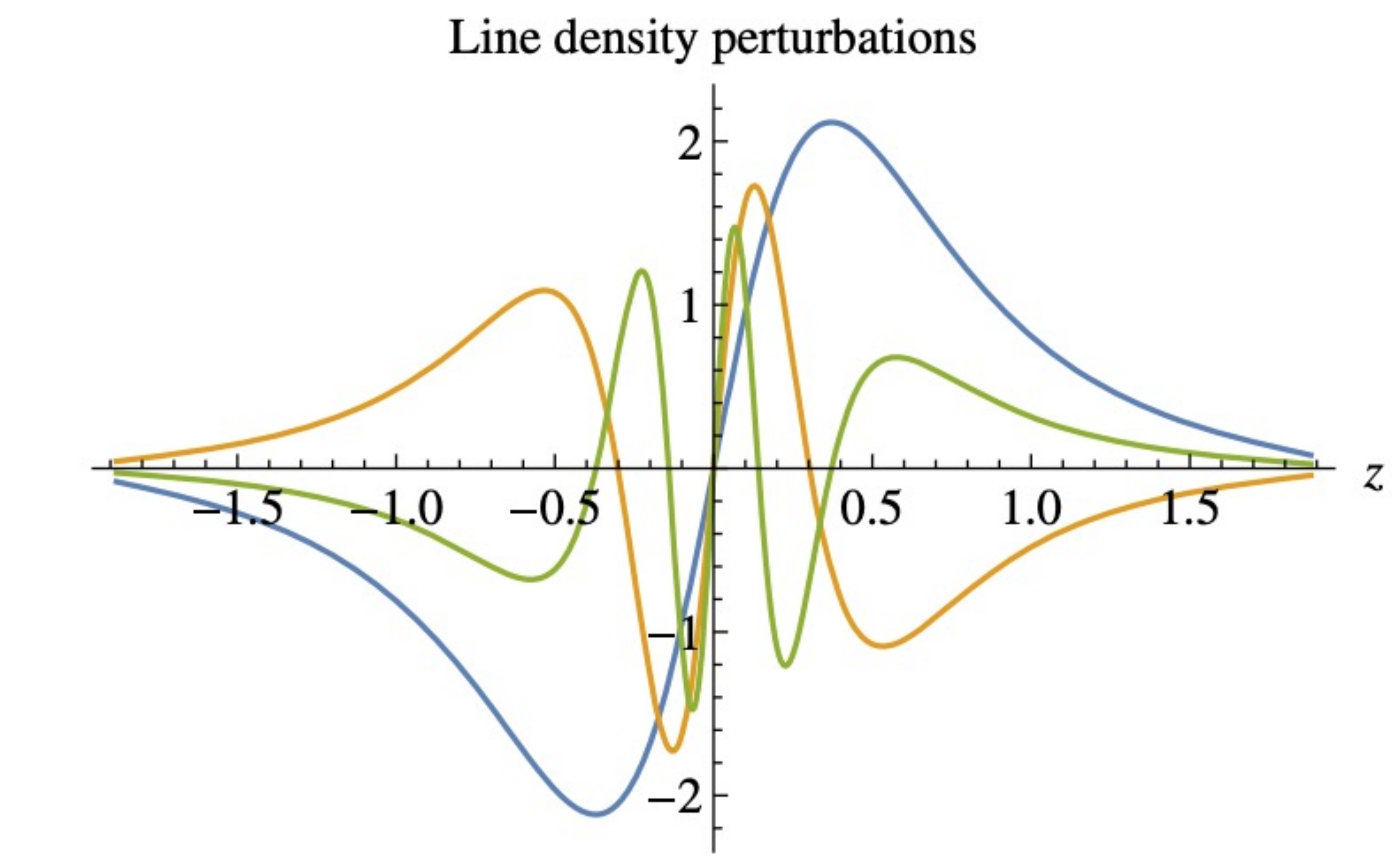


FIG. 4. Line density perturbations, Eq. (25), associated with the first three modes.

Comparison with the KAS article

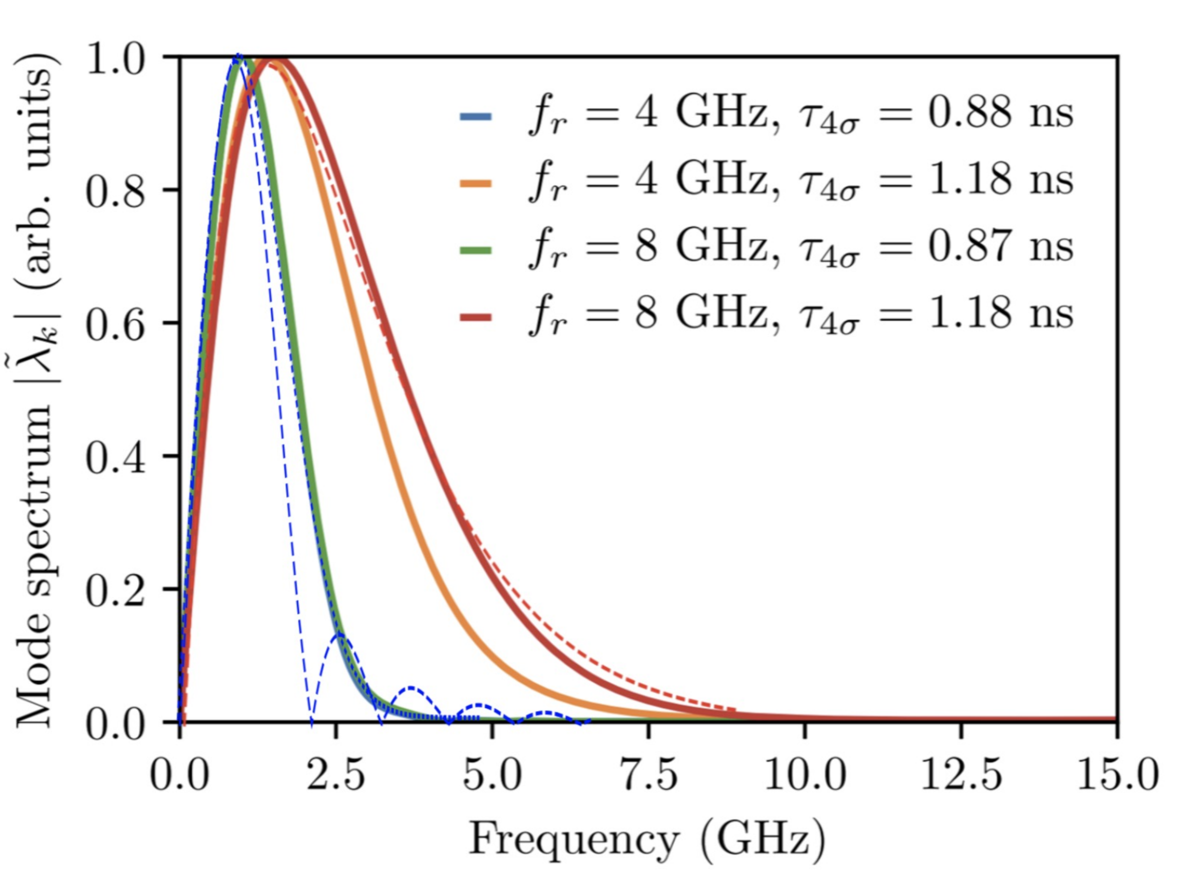


FIG. 6. Fourier transforms of the leading dipole mode line density. The general frame with its solid lines is a copy of Fig. 11, right, of Ref. [8] for the specified LHC beam parameters. The dashed red line shows the Fourier transform of the parameter-less leading mode profile, Fig. 4, scaled to the solid red line parameters. The blue dashed line shows the transform of the line density perturbation for the solid green and blue line parameters, if it corresponded to the rigid-bunch motion. The blue dotted line, practically coinciding with almost identical green and blue solid lines, is a fit of a Gaussian rigid-bunch mode.

Coupled-bunch (CB) growth rates

$$[\omega - \Omega(I)]f(I) = -F'(I)\int_0^{\hat{I}} K(I,I')\,f(I')\mathrm{d}I' - 2\pi\Gamma\,F'(I)\sqrt{I}\int_0^{\hat{I}} f(I')\sqrt{I'}\,\mathrm{d}I'$$

with the CB term only, it reduces to the Sacherer dispersion relation,

$$-2\pi\Gamma\int_0^{\hat{I}}\frac{F'(I)\,I}{\omega-\Omega(I)}\,\mathrm{d}I = 1$$

If the CB tune shift is small compared with SB, then

$$\delta\omega_{\mathrm{cb}} = \pi\Gamma\,|F'|\,\alpha^4\left(\int_0^\infty \Phi_\beta(r)r^2\mathrm{d}r\right)^2 = 16\pi\,\Gamma k^4\frac{|F'|^5}{\Omega'^4}\left(\int_0^\infty \Phi_\beta(r)r^2\mathrm{d}r\right)^2 \quad \propto\ N^5$$

$$\delta\omega_{\mathrm{cb}} = \Gamma\,C_\beta\Delta z_\beta$$

Higher multipoles

$$[\omega - m\Omega(I)]f(I) = -mF'(I)\int \mathrm{d}I'\,[K_{\rm sb}(I,I')_m + K_{\rm cb}(I,I')_m]\,f(I')$$

$$K(I,I')_m = -\frac{2}{\pi}\int_0^\pi \mathrm{d}\phi\cos m\phi\int_0^\pi \mathrm{d}\phi'\cos m\phi'\,W(z(I,\phi)-z(I',\phi'))$$

$$\omega = m(\Omega(0)+\Delta\omega)$$

$$(\Delta\omega + |\Omega'|I)f(I) = |F'|\int \mathrm{d}I'\,[K_{\rm sb}(I,I')_m + K_{\rm cb}(I,I')_m]\,f(I')$$

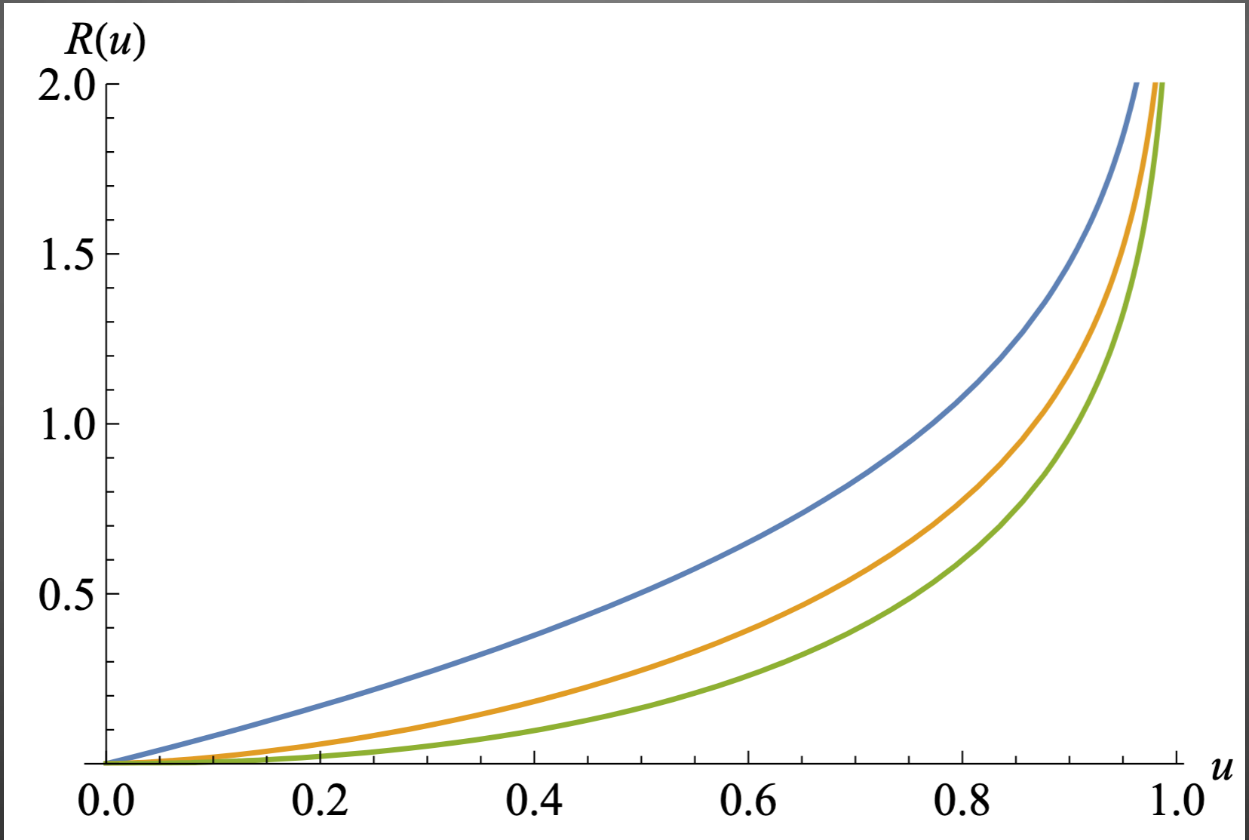
$$K_{\rm sb}(I,I')_m = 2k\int_0^\infty \mathrm{d}q\,\mathrm{J_m}(q\,b)\,\mathrm{J_m}(q\,b')$$

Kernels

$$K_{\rm sb}(I,I')_1 = \frac{4k}{\pi b_{\rm min}}\,[\mathrm{K}(u)-\mathrm{E}(u)] \equiv \frac{4k}{\pi b_{\rm min}}\,R_1(u);$$

$$K_{\rm sb}(I,I')_2 = \frac{4k}{\pi b_{\rm min}}\,\frac{(u+2)\mathrm{K}(u)-2(u+1)\mathrm{E}(u)}{3\sqrt{u}} \equiv \frac{4k}{\pi b_{\rm min}}\,R_2(u);$$

$$K_{\rm sb}(I,I')_3 = \frac{4k}{\pi b_{\rm min}}\,\frac{(4u^2+3u+8)\mathrm{K}(u)-(8u^2+7u+8)\mathrm{E}(u)}{15u} \equiv \frac{4k}{\pi b_{\rm min}}\,R_3(u)$$



SB Quadrupole Eigenvalues

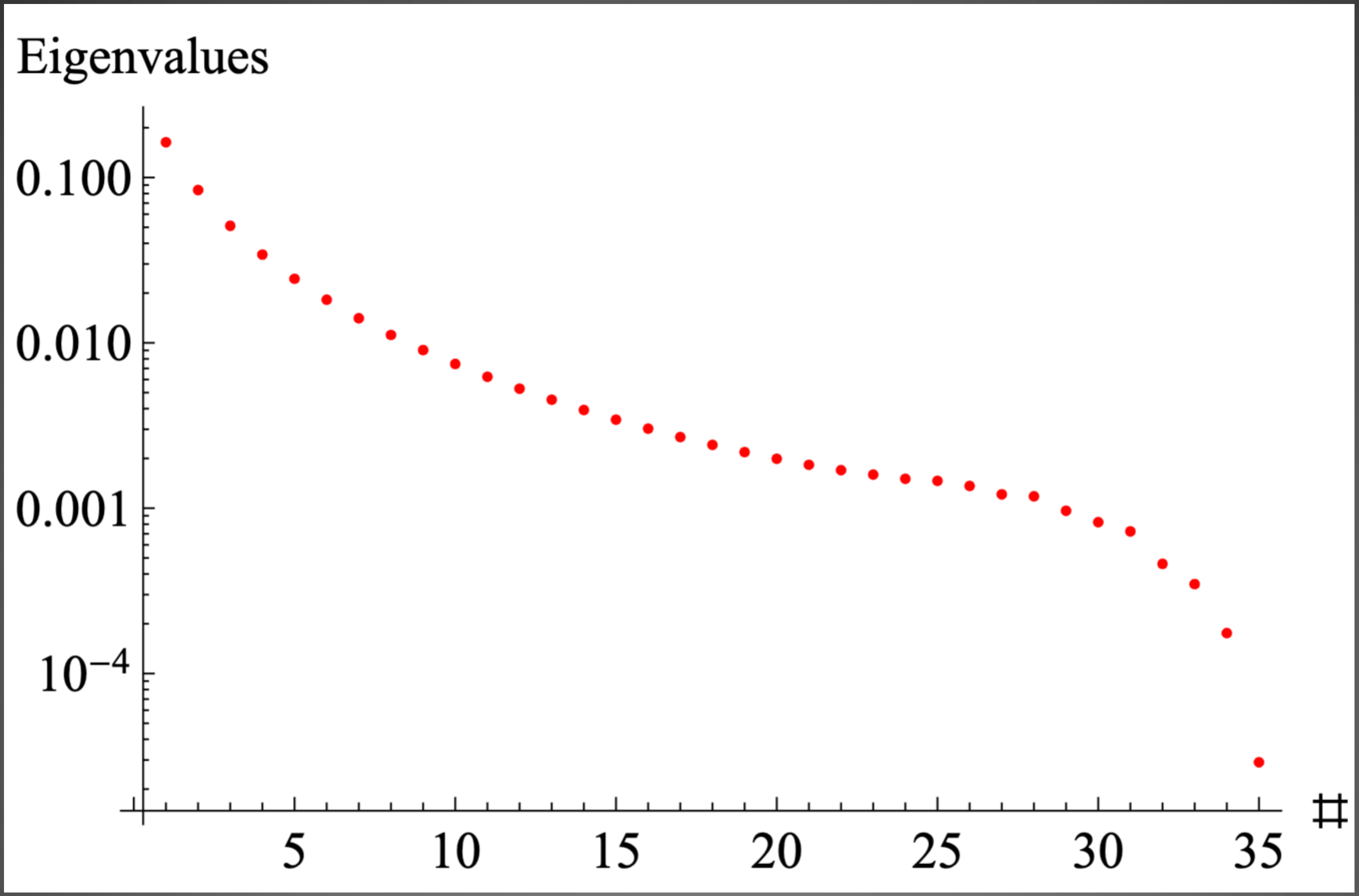


FIG. 8. Eigenvalues for the quadrupole modes. For the leading mode, $\nu = 0.16$.

Quadrupole Eigenfunctions

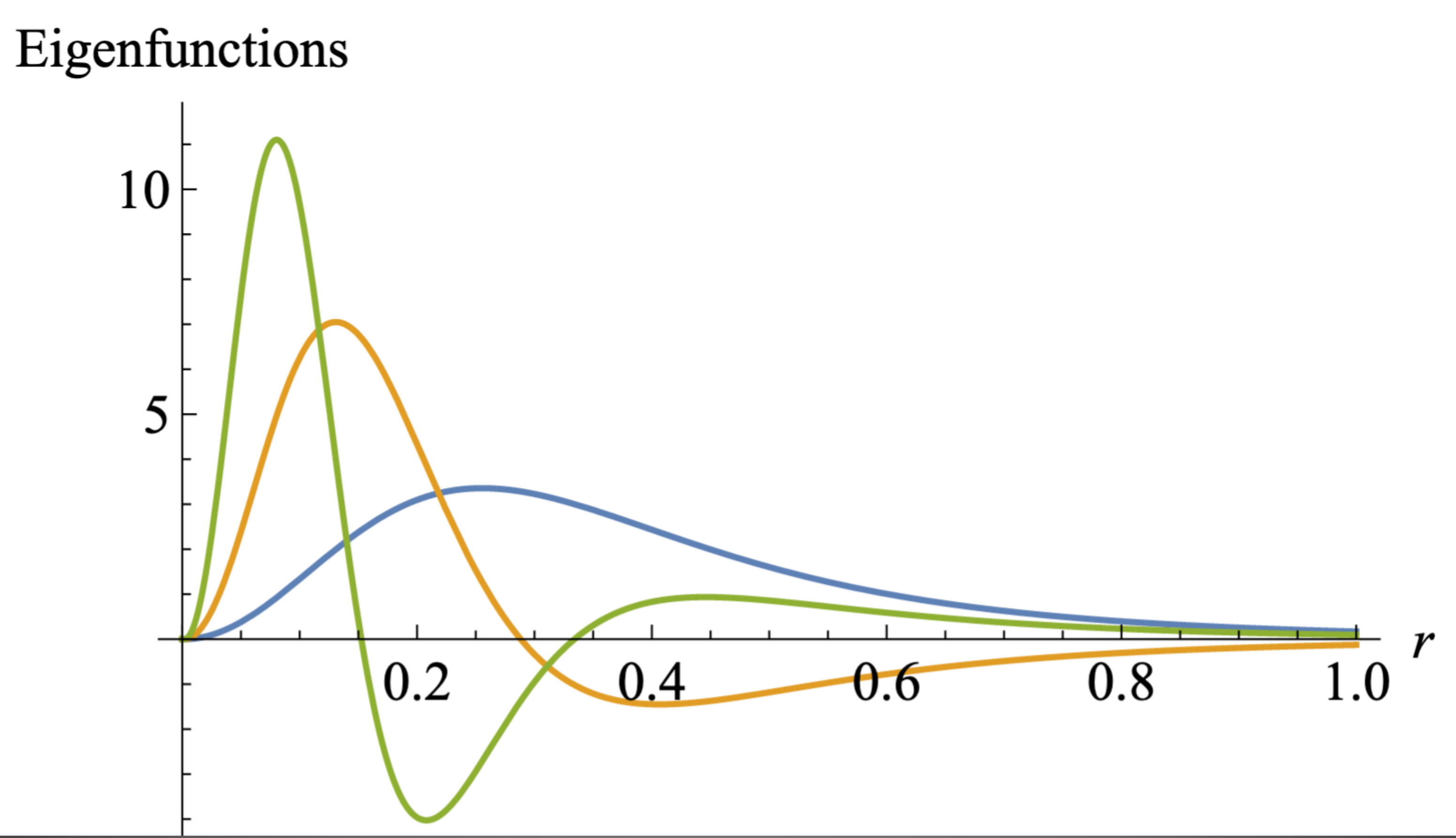


FIG. 9. Eigenfunctions for the 1st, 2nd and 3rd discrete quadrupole modes.

At small arguments, $\Phi_{m\beta}(r) \propto r^m$

Quadrupole mode line density

$$\rho(z) = \int \mathrm{d}p f(I) \cos(2m\phi) = 2\int_{|z|}^{\infty} \mathrm{d}r \frac{\Phi(r)}{\sqrt{r^2 - z^2}} (1 - 2z^2/r^2)$$

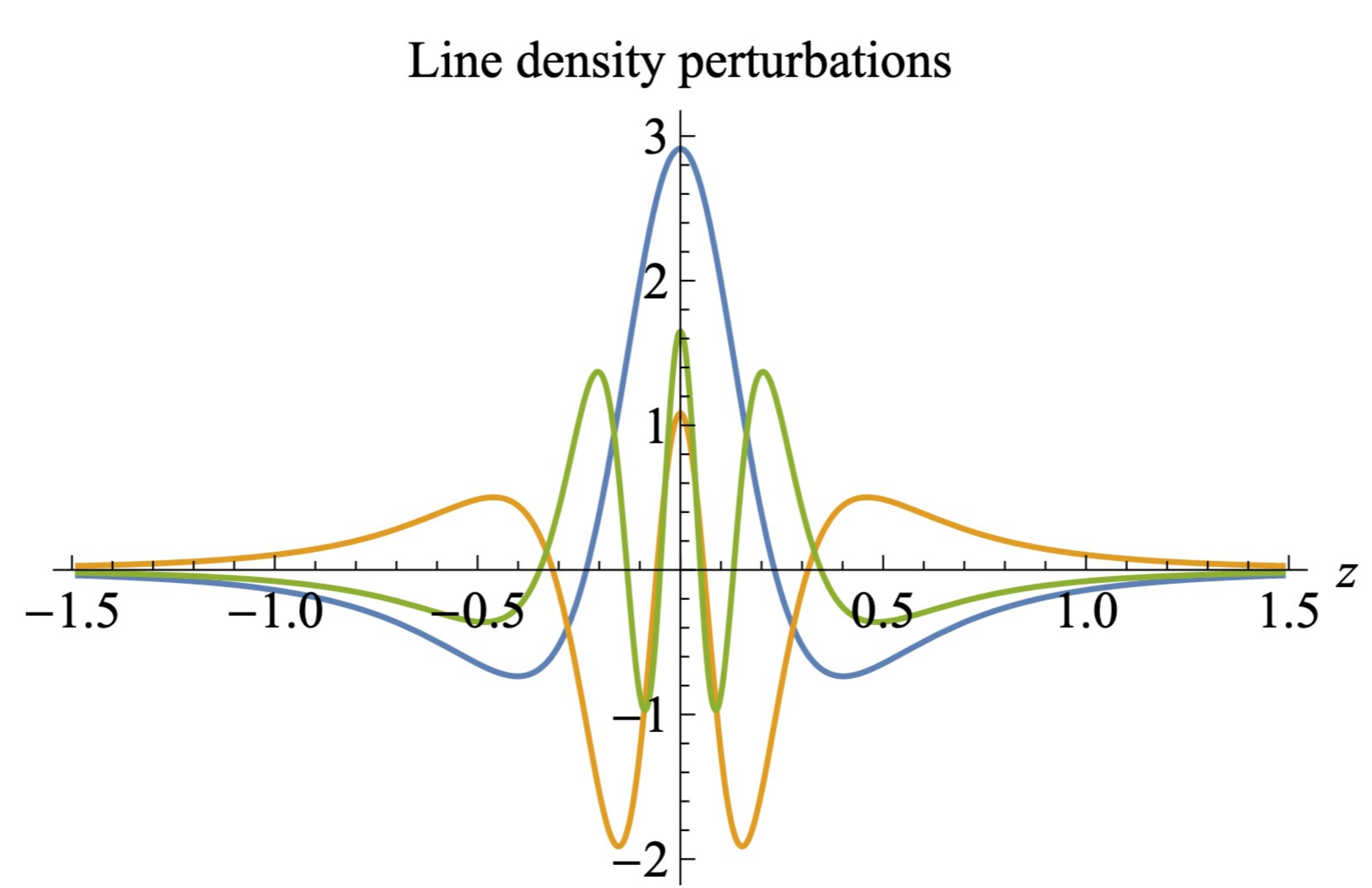


FIG. 10. Line density perturbations for the first, second and third discrete quadrupole modes.

CB growth rates for arbitrary multipolarity *m*

If the CB term is smaller than the SB, then the perturbation theory results in

$$\delta\omega_{\mu m\beta} = |F'|\tilde{W}_{\mu m}\alpha^{2m+2}\left(\int_0^\infty \mathrm{d}r\, r^{m+1}\Phi_{m\beta}(r)\right)^2 . \quad \propto N^{2m+3}$$

$$W_{\mathrm{cb}} \to \frac{N r_0 \eta\, \omega_{\mathrm{rf}}^2}{\gamma C_0\, \Omega_0^2} W_{\mathrm{cb}}$$

For the dipole modes,

$$\Gamma = \frac{N r_0 \eta\, c^2}{2\gamma C_0\, \Omega_0} \sum_{n=1}^{\infty} W''(-n s_{bb}) \exp\left(i n \psi_\mu\right)$$

the CB term dominates over the SB one, Sacherer–type dispersion relation follows:

$$-2^m \tilde{W}_{\mu m} \int_0^{\hat{I}} \frac{F'(I) I^m}{\Delta\omega + |\Omega'| I} \mathrm{d}I = 1 .$$

What is new here?

1. Zero LLD threshold for repulsive inductance is explained *on fingers*.
2. *Four SC regimes* are suggested to distinguish.
3. For *weak SC*, the eigen–problem is reduced to a parameter–less equation.
4. Single–bunch spectra are found for arbitrary multipoles.
5. Coupled–bunch growth rates are presented analytically.
6. LLD+CB should result in persistent oscillations, where CB growth is saturated by nonlinearity, unless it is IBS–damped.

Summary slide, 5th ICFA mini-workshop on Space Charge

Theme: Bridging the gap in space charge dynamics

In 1-2 sentences, summarize the content of this presentation

Theory of longitudinal modes with weak SC is developed for SB and CB cases.

From your perspective, where is the gap regarding space charge effects?

Convective instabilities with chromaticity, damper and LD.

What is needed to bridge this gap?

Sufficient number of adequate people working on that.

Many thanks!